Application of an Optimal Stochastic Newton-Raphson Technique to Triangulation-Based Localization Systems

Khaled Kamal Saab
Department of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA, USA
khaled.saab.95@gmail.com

Samer Said Saab Jr.
Department of Electrical and Computer Engineering
Pennsylvania State University
State College, PA, USA
samer.saab.1st@gmail.com

***Abstract*— Trilateration-based localization schemes typically extract the needed information, such as distances between transmitters and receivers, from noisy received signal strength (RSS) measurements specific to the employed technologies. Such applications employ non-stochastic methods without appropriately targeting noise. This paper proposes employing a recent optimal stochastic Newton-Raphson (NR) algorithm with measurement noise rejection capability for a class of localization applications. In order to show the effectiveness of this algorithm, we numerically consider an anchor-free localization problem with random initial guess. Numerical results show that the proposed recursive algorithm provides significant improvement over the traditional NR method.**

Keywords*— *Newton's method; localization; relative location; anchor free; stochastic optimization

I. INTRODUCTION

On stochastic optimization

Many engineering applications are based on an optimization setting where only noisy estimates of the objective function and/or its gradient are available. One class of applications includes localization systems that are based on triangulation techniques. Triangulation techniques include lateration methods, which determine the distance between the transmitter and receiver; and angulation methods, which determine the angle between the transmitter and the main axis of the receiver's antenna. In order to compute distances and angles, methods such as received signal strength (RSS), time of arrival (TOA), time difference of arrival (TDOA), phase of arrival (POA), and angle of arrival (AOA) are used. For example, trilateration-based positioning schemes extract the needed information, such as distances between transmitters and receivers, from noisy RSS measurements specific to the employed technologies; e.g., light-emitting diode [1] and radio-frequency identification [2]. Radar ranging data are also employed for the localization of planes [3]. Such applications employ non-stochastic methods such as the traditional Newton-Raphon (NR) for localization based on noisy ranging measurements. However, standard optimization methods, such as NR, steepest descent, simulated annealing, or genetic algorithms, can lead to non-convergent and highly suboptimal solutions without appropriately treating noise [4].

Many methods have been proposed (see, e.g., [4]) to deal with noisy function measurements. Stochastic approximation algorithm provides a simple and effective approach for finding root or minimum of nonlinear function whose evaluations are contaminated with noise. One straight forward and efficient approach is gradient descent such as the Robbins-Monro algorithm [5] and its derivatives. However, gradient-based algorithms generally suffer from relatively slow convergence in plateau regions and they do not necessarily converge when the objective function is steep [6]. On the other hand, the NR algorithm achieves quadratically fast convergence [7]. However, the traditional NR algorithm does not incorporate any scaling associated with the statistics of noisy measurements. The proposed stochastic NR algorithm proposed in [8] deals with the latter where the basic problem of interest is the root-finding problem under noisy measurements of the objective function, $f(z)$, where $z \in \mathbb{R}^N$, and $f(z) \in \mathbb{R}^M$ where $M \geq N$. The work in [8] proposes a recursive algorithm providing optimal iterative-varying gains associated with the NR method. The development of the proposed optimal algorithm is based on minimizing a stochastic performance index. The estimation error covariance matrix is shown to converge to zero for linearized functions while considering additive zero-mean white noise.

On anchor-free localization

Employing large-scale networks using efficient low-cost sensors is gaining popularity in many areas, such as tracking, surveillance, environmental, civil, and military applications. Equipping each sensor with a global positioning system is one method of locating each sensor, but it is also costly and inadequate in some situations where there is no network coverage. Some sensor network applications, such as shape recognition [9], or augmented reality, could benefit from anchor-free localization where relative positions of the sensors are known using internode measurements without using the global reference. A number of techniques can be used to calculate internode measurements, such as the distance between nodes using connectivity, RSS, time-of-arrival, time-difference-of-arrival, or frequency-difference-of-arrival [9]–[15]. Once the distances among nodes are known, different methods can be used to solve for their relative positions, such as triangulation [16], the iterative Gauss-Newton method [17], linear estimators [18], steepest descent method [19], and weighted least-squares solutions [20]. In [16], a cooperative algorithm is suggested,

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where distances are used to compute angles between nodes. In [17], a distributed Gauss-Newton method is proposed with weighted averaging, where a good initial guess is needed for the convergence of the solution. In [18], a Newton-Raphson method is proposed, where linear estimators are used as an initial guess. In [19], a steepest descent method is used to solve for nonlinear equations that relies on accuracy of initial conditions. Other effective methods are also applied to the cooperative localization problem using a small number of anchors. In [21] nonlinear Gauss-Seidel method is employed, in [22] the approach is based on a weighted least square-squares, [23] employs conjugate gradient descent, and [24]-[25] are based on nonconvex gradient optimization.

In this paper, we propose employing an optimal stochastic Newton-Raphson algorithm [8] for localization applications. As a localization application, we consider an anchor-free localization problem with random initial guess. In anchor free relative localization problem, the distances between any two nodes are used to set up a set of nonlinear equations. In the case where (x, y) coordinates of each node are the unknowns and n nodes are considered, then the number of unknowns becomes $2n$ and the number of equations $n(n-1)/2$, so the number of equations increases more rapidly than the number of unknowns and is greater for $n > 5$. The additional equations are considered as they provide additional information about the relative positions of the nodes. The recursive algorithm under consideration utilizes the statistical measurement error model, which can be in terms of the distances (e.g., RSS measurements [1]-[2]), in order to construct the time-varying gains while minimizing the variance of errors. We show that the proposed recursive algorithm provides significant improvement over the NR method.

The paper is organized as follows. Section II summarizes the main results presented in [8]. Section III maps noisy trilateration-based localization to zero-finding problem and also includes application to anchor-free localization while comparing the performance of the NR method with the one in [8]. Concluding remarks are presented in Section IV.

II. Background on Stochastic Newton-Raphson Method [8]

This section summarizes the results presented in [8].

A zero-finding problem gives M functional relations to be zeroed, that is,

$$f(z) = 0 \tag{1}$$

where $z \in \mathbb{R}^N$, and $f(.) \in \mathbb{R}^M$. The assumption in the zero-finding setting is that $f(.)$ is not available directly, but must be estimated through a noisy estimate of $f(.)$, $\hat{f}$. In this letter, we consider noisy observations of $f(.)$ consisting of additive noise, that is, $\hat{f} = f(.) + \epsilon$, and consider unconstrained optimization and the case where during each iteration, one measurement of $f(.)$ is available. We consider the following setting at iterative instant, k:

$$\hat{f}(.) = f(.) + g(k)v(k) \tag{2}$$

where $v(k) \in \mathbb{R}^M$ a zero-mean white random process, $g(k) \in \mathbb{R}^{M\times M}$ is a deterministic function.

In the neighborhood of $\hat{z} \in \mathbb{R}^N$, while assuming that the elements of f are continuously differentiable, f can be expanded in Taylor series as follows:

$$f(\hat{z} + \Delta z) = f(\hat{z}) + J(k)\Delta z + O(\Delta z^2)$$

where the elements of the Jacobian matrix, $J(.) \in \mathbb{R}^{M\times N}$, are defined as $J_{ji}(k) \triangleq \frac{\partial f_j}{\partial z_i}\Big|_{\hat{z}=\hat{z}(k)}$. In what follows, we neglect terms of order Δz^2 and higher leading to

$$f(\hat{z} + \Delta z) = f(\hat{z}) + J(k)\Delta z \tag{3}$$

Assuming that $J(k)$ is full-column rank, the iterative method proposed in [8] is given by

$$\hat{z}(k+1) = \hat{z}(k) - K(k)J^{\dagger}(k)\hat{f}(\hat{z}(k)) \tag{4}$$

where $K(k) \in \mathbb{R}^{N\times N}$ is a multiplicative gain, and $J^{\dagger}(k) = [J^T(k)J(k)]^{-1}J^T(k)$. The problem is to find $K(k)$ such that the covariance of $\delta z \triangleq z - \hat{z}$ is minimized at each time instant.

We denote by the error covariance matrix, $P(k) \triangleq E[\delta z(k)\delta z^T(k)]$, $F(k) \triangleq J^{\dagger}(k)g(k)$ and $R(k) \triangleq E[v(k)v^T(k)]$, where $E[.]$ is the expectation operator.

Proposition 1 [8]. Assume that $R > 0$ and F is full rank. If $\lim_{k\to\infty} P(k) = 0$, then it is necessary to have $\lim_{k\to\infty} K(k) = 0$.

It is important to note that if we set $K(k) \equiv I, \forall k$, then (4) reduces to the traditional NR method. Proposition 1 implies that the traditional NR method cannot drive the error covariance matrix to zero since $\lim_{k\to\infty} K(k) = 0$ is a necessary condition. The latter requires an iterative-varying gain, $K(k)$.

The following theorem presents an optimal recursive algorithm for generating the gain, $K(k)$, at each instant, k.

Theorem 1 [8]. Consider the linear vector function given in (3) and the method proposed in (4). Assume that there exists a z such that $f(z) = 0$, and the Jacobian matrix is full-column rank, $k \geq 0$. The gain $K(k)$ that minimizes the mean-square of $\delta z(k) \triangleq z - \hat{z}(k)$ at each k^{th} instant is given in the following recursive formulas for all $k > 0$,

$$K(k) = P(k)(P(k) + F(k)R(k)F^T(k))^{-1} \tag{5}$$

$$P(k+1) = (I - K(k))P(k) \tag{6}$$

The following theorem shows that optimal recursive algorithm (5) and (6) drives the error covariance matrix to zero while satisfying the necessary condition given in Proposition 1.

Theorem 2 [8]. Consider the recursive algorithm presented in (5) and (6). If $P(0) > 0$ and $F(k)R(k)F^T(k) > 0, \forall k \geq 0$, then $0 < \lambda(I - K(k)) < 1$, $\lim_{k\to\infty} P(k) = 0$, and $\lim_{k\to\infty} K(k) = 0$, where $\lambda(M)$ denotes the eigenvalues of M.

Remark 1 [8]. The results of Theorems 1 and 2 assume a linear system of functions without considering modeling errors due to linearization. Consequently, $\lim_{k\to\infty} P_k$ may not be zero. For example, if errors due to linearization are added to (3) and modelled as additive zero-mean white noise with covariance $Q_k \geq 0$, then it can be shown that the optimal gain remains $K_k = P_k(P_k + FRF^T)^{-1}$ and the associated covariance becomes $P_{k+1} = (I - K)P_k + Q_k$, which is always bounded; however, $\lim_{k\to\infty} P_k \neq 0$. Such a scenario is not considered in the proposed algorithm since the modelling part of such errors is function specific. In some cases of nonlinearities, P_k and K_k may converge to zero too early. In order to remedy this problem, we reset $P_{k+1} \equiv (I - K)P_k + Q_k$ for an arbitrary Q_k after a couple of iterations.

III. Application to Trilateration-based Localization

A. Trilateration-based localization and the zero-finding Problem

This section maps a trilateration-based localization problem to the zero-finding problem described in (1) and (2) and describes application of the iterative algorithm (4). The range equation is given by

$$h(z^*) = d$$

where d is the range between two points and $h(z)$ is the corresponding range equation, which is known to the user. For example, if z denotes the two-dimensional Cartesian coordinates (x, y), then

$$h(z) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d \quad (7)$$

where $(x_i, y_i), i \in \{1,2\}$ are the coordinates of the two points under consideration. The problem becomes finding the root, z^*, of $f(z^*) \triangleq h(z^*) - d = 0$ associated with (1). However, the range is usually measured with some errors; e.g., $\hat{d}(k) = d + \varepsilon_1(k)$ at different discrete-time instants, k, where $\varepsilon_1(k)$ represents the measurement errors, which could depend on z^*. It is implicitly assumed that the positions of the points to be localized are motionless and each time instant, k, we have an measurement of the distance with a random measurement error, $\varepsilon_1(k)$. Therefore, based on the available measurements, we have

$$\tilde{f}(z^*) = h(z^*) - d - \varepsilon_1(k)$$

or

$$\tilde{f}(z^*) = f(z^*) - \varepsilon_1(k)$$

Next, we consider the iterative algorithm (4) where $\hat{f}(\hat{z}(k)) = f(\hat{z}(k)) + g(k)v(k)$ and $f(\hat{z}(k)) = h(\hat{z}(k)) - d$. Therefore,

$$\hat{f}(\hat{z}(k)) = h(\hat{z}(k)) - d + g(k)v(k) \quad (8)$$

Consequently, while iterating, we can use the different measurements of d, $\hat{d}(k) = d + \varepsilon_1(k)$. Thus, $g(k)v(k) = -\varepsilon_1(k)$, or $\hat{f}(\hat{z}(k)) = f(\hat{z}(k)) - \varepsilon_1(k) = h(\hat{z}(k)) - d - \varepsilon_1(k)$. That is, at k, we have $\hat{z}(k)$ and we evaluate $h(\hat{z}(k))$ then we subtract, from the latter, the measurements, $d + \varepsilon_1(k)$, in order to get

$$\hat{f}(\hat{z}(k)) = f(\hat{z}(k)) - \varepsilon_1(k) \quad (9)$$

Example

In this example we consider the problem setup represented by (7)-(9) while studying the performance of the proposed method and the NR method for the following cases:

a) Noise with different distributions, in particular, zero-mean Gaussian noise with standard deviation equals to σ and uniform distribution on the interval $[-0.5\sigma\sqrt{12}, 0.5\sigma\sqrt{12}]$ so that the noise power for both distributions are identical.

b) Employment of observation averaging, that is, $\tilde{d}(k) = \frac{1}{L}\sum_1^L \hat{d}(i)$. For this case we consider the noise, $n_i \in \aleph(0, \sigma)$.

In this example, we consider the setup depicted in Fig. 1.

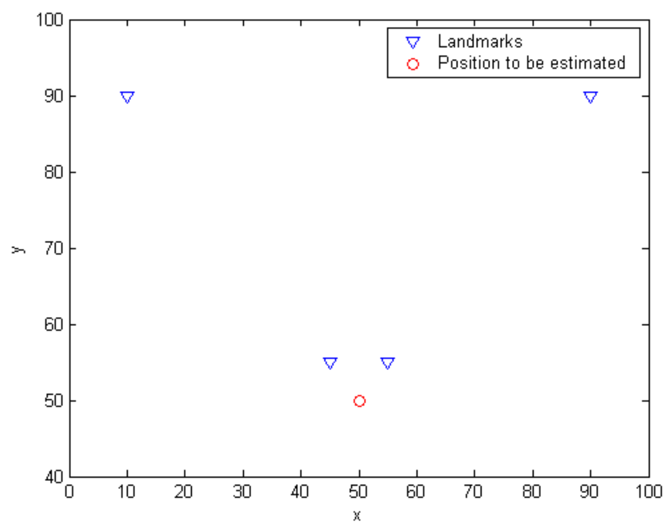


Fig. 1. Cartesian coordinates of the known landmarks and the object position to be estimated

To estimate (x_P, y_P), using the noisy measurement of distances, $\hat{d}_i(k)$, between landmarks and object, given by $\hat{d}_i(k) = d_i + n_i(k)d_i, i \in \{1,4\}$, where the coordinates of d_i are (x_i, y_i) and $n_i(k)$ is the noise under consideration.

In the case where no averaging is considered, we set the measurement error covariance matrix: $\hat{R}(k) \triangleq \sigma^2 I$, $\hat{d}_i(k) \triangleq \left\| \begin{matrix} x_i(k) - \hat{x}_P(k) \\ y_i(k) - \hat{y}_P(k) \end{matrix} \right\|$, $g_{ii}(k) \equiv \hat{d}_i(k)$ and $g_{i\neq j}(k) = 0$.

In the case of averaging, we set the measurement error covariance matrix: $\hat{R}_{\text{averaging}}(k) = \frac{1}{k}\hat{R}(k)$.

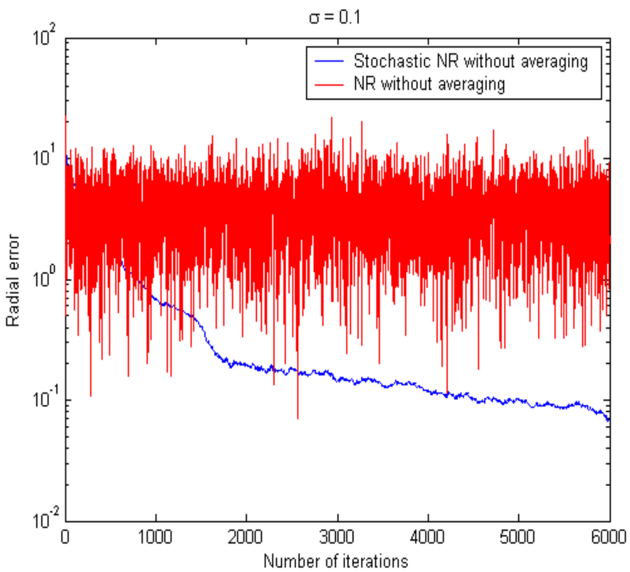
For Case a) where we consider noise with different distributions, we use $\sigma = 0.1$, and for each run we use 6,000 iterations, and conduct 100 independent runs for each experiment and report the average value of radial errors and their standard deviations. One sample is shown is Fig. 2 (i). The performance results are summarized in Table 1.

Examining Table 1, we find the results are almost identical when considering different distributions; however, the radial errors (average) corresponding to the proposed method are 44 times smaller than the traditional NR method with much better stability between different runs (standard deviation).

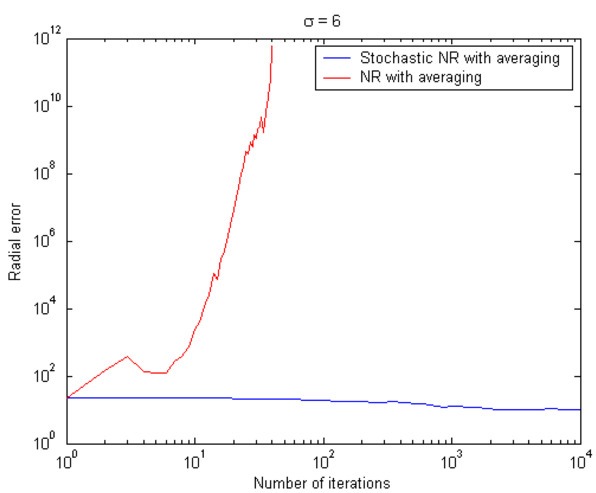
TABLE 1. PERFORMANCE RESULTS OF THE MONTE CARLO SIMULATIONS: RADIAL ERRORS WITH $\sigma = 0.1$.

	Average		Standard Deviation	
Noise	NR	Proposed	NR	Proposed
Gaussian	4.45	0.10	2.33	0.04
Uniform	4.50	0.10	2.12	0.04

For Case b) where we consider averaging the observations, we use $\sigma = 6$. Fig. 2 (ii) shows a sample of the performance of the proposed method and the traditional NR method. For relatively large values of σ the traditional NR method diverges whereas after 6,000 iterations the radial error decreases to 1, which is significantly smaller than the measurement errors corresponding to $\sigma = 6$.



(i)



(ii)

Fig. 2. Performance of the proposed method and the NR method: (i) without averaging and (ii) with averaging

B. Application to anchor-free localization

In this section we numerically apply the proposed algorithm to an anchor-free localization problem and compare performance with the traditional NR method, which is similar to the algorithm in (4) except for setting $K(k) = I, \forall k$. In this example we only consider the relative localization, under the following assumptions:

a) Each sensor (or node) can measure the distances to all other sensors in two dimensions, x and y.
b) The two-dimensional locations of twenty sensors are randomly generated. In particular, x and y are selected separately from uniformly distributed random variable with mean = 50m and standard deviation equals to $\sigma_{\text{location}} = \sqrt{10^4/12}$. That is, the sensors are randomly distributed in 100×100 m^2 area. The system is then composed of 189 nonlinear equations with 36 unknowns – excluding the two reference points.
c) The initial locations are generated randomly *and independently* of the true locations of the sensors.
d) Two reference sensors are randomly selected. In particular, we randomly select the first reference sensor. We select the second reference sensor to be the one furthest from the first reference point.
e) For convenience of presentation, we translate and rotate all points such that one reference sensor is set at the origin, and the other reference point is set at ($x = 0$, $y = d_{12}$), where d_{12} is the distance between the two reference sensors.
f) For the measurement errors, we add a scaled random error, Δd_{ij}, to each true distance, d_{ij}, between the i^{th} and j^{th} sensors given by: $\Delta d_{ij} = n_{ij} d_{ij}$, with $n_{ij} \in \aleph(0, \sigma)$. In particular, n_{ij} is a zero-mean statistically independent Gaussian random variable with standard deviation equals to σ. However, we consider the following two scenarios:

Scenario 1: We assume that we have continuous measurements of all distances, and at each iteration we acquire a new set of erroneous measurements, and use them as we iterate. That is, each iteration employs a new set of erroneous measurements

Scenario 2: We fix one set of measurements of the distances and iterate to estimate the relative locations of the sensors while keeping one set of measurements fixed throughout all iterations.

The nonlinear function under consideration is

$$(\hat{x}_i - \hat{x}_j)^2 + (\hat{y}_i - \hat{y}_j)^2 - \left(d_{ij} + n_{ij} d_{ij}\right)^2 = 0$$

We use the mean absolute error to assess the performance of the proposed algorithm and the NR method. We denote by $\text{MAE}_\xi \triangleq \text{AVG}_{i\in S}\left|\xi_i - \hat{\xi}_i(k)\right|, \xi \in \{x, y\}$, where AVG is the average operator, S is a set of relevant data spanning all n sensors excluding the two reference sensors. We also use the following metrics: $\overline{\text{MAE}}_\xi \triangleq \text{AVG}_{MC}(MAE_\xi)$, where MC includes different Monte Carlo runs, $\overline{\text{var}}_\xi \triangleq \text{AVG}_R(\text{var}_{i\in S}(\xi_i - \hat{\xi}_i(k)))$, where var is the variance operator, and $\overline{\text{AVG}}_\xi \triangleq \text{AVG}_R(\text{AVG}_{i\in S}(\xi_i - \hat{\xi}_i(k)))$. It is important to note that standard deviation of the measurement error relates to σ and can be closely estimated to be equal to 57.5σ. For all simulations, we use $P(0) = 1000$.

C. Setup and Simulation results: Scenario 1

Based on the guidelines presented in Remark 1, we implement (5) and (6), however, every 10 iterations, we use instead of (6), the following equation to update $P(k)$:

$$P(k+1) = \big(I - K(k)\big)P(k) + G(k)G^T(k) \qquad (10)$$

where $G(k) \triangleq J^{\dagger}(k)\hat{f}(\hat{z}(k))$. It is important to note that as $G(k)$ decreases the actual error covariance also decreases, since as $f(\hat{z}(k))$ decreases, the error decreases. For this Scenario, we use $\sigma = 0.1$ to generate the measurement errors as described in list item f) above. Fig. 3 and Fig. 4 illustrate the setup and performance for one random run. However, we simulate the same scenario for 250 different runs. The performance of the 250 runs is summarized in Table 2. The improvement of the proposed method over the NR method is shown to be about one order of magnitude. The mean absolute radial error is 0.6m for the proposed method and 4.66m for the NR method, in a 10,000m^2 area, where the standard deviation of measurement errors is about 5.75m.

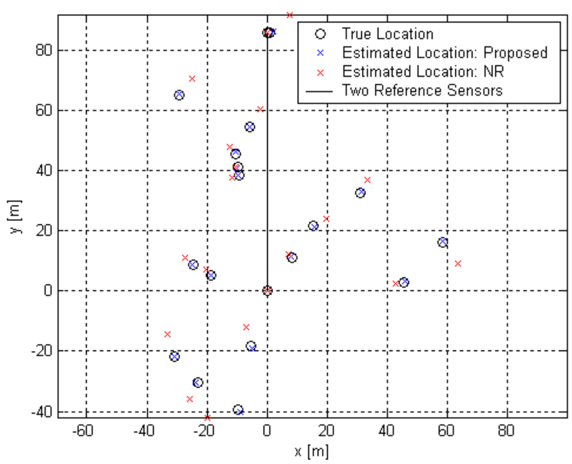


Fig. 3. Scenario 1: A sample of the sensors space and two-dimensional performance.

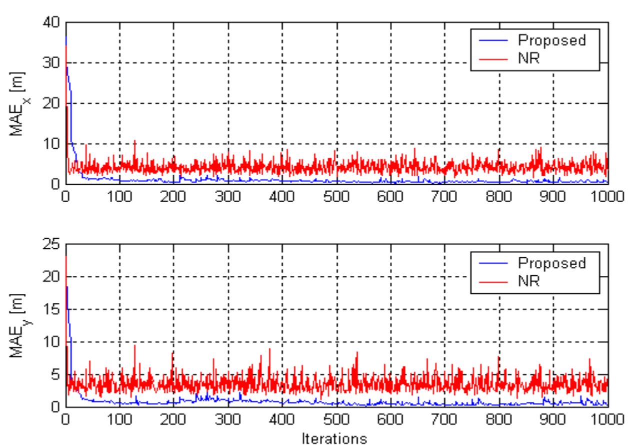


Fig. 4. Scenario 1: Performance corresponding to the case depicted in Fig. 3.

TABLE 2. PERFORMANCE RESULTS OF THE MONTE CARLO SIMULATIONS: SCENARIO 1.

	$\overline{\text{MAE}}_x$	$\overline{\text{MAE}}_y$	$\overline{\text{var}}_x$	$\overline{\text{var}}_y$	$\overline{\text{AVG}}_x$	$\overline{\text{AVG}}_y$
NR	3.45	3.14	13.3	12.8	0.05	0.13
Prop.	0.45	0.4	1.74	0.63	0.05	0.11

D. Setup and Simulation results: Scenario 2

Instead of (10), we also use (10) to update $P(k)$ at every iteration. The simulations involve varying the value of $\sigma \in \{0.1, 0.2, \ldots, 1\}$ and run Monte Carlo simulations for each value. Plots of mean absolute errors are included in Fig. 5 while comparing them to the standard deviation of measurement errors. By examining Fig. 5, we note that for $\sigma < 0.5$, the performance of the proposed method and the NR method are almost identical. However, the superiority of the proposed method becomes more evident for larger values of σ, $\sigma \geq 0.5$. For example, when $\sigma = 1$, the radial error of the NR method is 60m whereas the radial error corresponding to the proposed method is 50m.

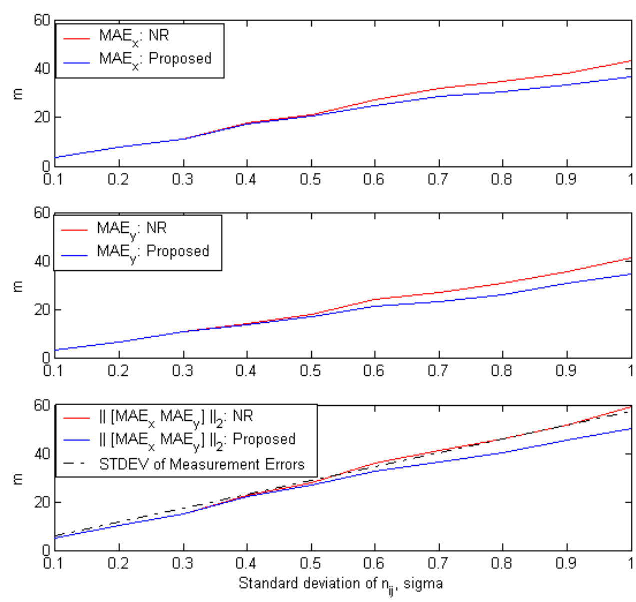


Fig. 5. Scenario 2: Performance pertaining to different scales of the measurement errors.

Remark 5. Unlike Scenario 1, Scenario 2 assumes that n_{ij} is constant for all iterations. However, the proposed algorithm assumes n_{ij} to be zero-mean white noise as in the case of Scenario 1. Consequently, the performance results pertaining to Scenario 1 are much better.

Remark 6. The error in the y-dimension is smaller than the one in the x-dimension, which is due to the fact that the sensor corresponding to the largest value of y, with the largest expected error, is used as a reference point $\notin S$.

Remark 7. When the number of sensors is increased, the resulting errors consistently decrease since the number of equations increases at a faster rate than the number of unknowns.

IV. Conclusion

This paper proposed applying a recent stochastic algorithm [8] to trilateration-based localization schemes. This methodology was applied to an anchor free localization problem with random initial condition. It was numerically demonstrated that in case where: a) 20 sensors are considered and uniformly distributed over 10,000m^2 area where each sensor can measure the distances to all other sensors, and b) continuous measurements of distances among sensors are available with standard deviation of additive measurement errors of 5.75m, the resulting radial error of the proposed approach is 0.6m whereas the radial error corresponding to the traditional NR is 4.66m. In addition, the method in [8] showed better performance over the NR method when measurement errors are held to a constant throughout the iterative process. Performance analysis in presence of different classes of measurement errors is left for future work.

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